

STABILITY OF RIGIDLY ROTATING RELATIVISTIC STARS WITH SOFT EQUATIONS OF STATE AGAINST GRAVITATIONAL COLLAPSE

MASARU SHIBATA¹

¹ Graduate School of Arts and Sciences, University of Tokyo,
 Komaba, Meguro, Tokyo 153-8902, Japan
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ABSTRACT

We study secular stability against a quasi-radial oscillation for rigidly rotating stars with soft equations of state in general relativity. The polytropic equations of state with polytropic index n between 3 and 3.05 are adopted for modeling the rotating stars. The stability is determined in terms of the turning-point method. It is found that (i) for $n \gtrsim 3.04$, all the rigidly rotating stars are unstable against the quasi-radial oscillation and (ii) for $n \gtrsim 3.01$, the nondimensional angular momentum parameter $q \equiv cJ/GM^2$ (where J , M , G , and c denote the angular momentum, the gravitational mass, the gravitational constant, and the speed of light, respectively) for all marginally stable rotating stars is larger than unity. A semi-analytic calculation is also performed, and good agreement with the numerical results is confirmed. The final outcome after axisymmetric gravitational collapse of rigidly rotating and marginally stable massive stars with $q > 1$ is predicted, assuming that the rest-mass distribution as a function of the specific angular momentum is preserved and that the pressure never halt the collapse. It is found that even for $1 < q \lesssim 2.5$, a black hole may be formed as a result of the collapse, but for $q \gtrsim 2.5$, the significant angular momentum will prevent the direct formation of a black hole.

Subject headings: black hole physics – relativity – hydrodynamics – stars: rotation

1. INTRODUCTION

At the final stage of their evolution, massive stars with initial mass $\gtrsim 8M_\odot$ form an iron core. Because of high temperature and density, the photodissociation of the iron to lighter elements proceeds rapidly (Shapiro & Teukolsky 1983). In the photodissociation, the thermal pressure is consumed, decreasing the adiabatic index Γ below $4/3$. Consequently, the core is destabilized, resulting in collapse to a neutron star or a black hole.

Recent numerical simulations of the collapse of primordial molecular clouds have suggested that the first generation of stars contains many massive members of mass $\sim 100\text{--}1000M_\odot$ (Bromm et al. 1999; Abel et al. 2000; Nakamura & Umemura 2001). Such massive stars form large helium cores that reach carbon ignition. It is known that after helium burning, cores of such stars encounter the electron-positron pair creation instability, bringing Γ below $\sim 4/3$ and triggering the collapse (Barkat et al. 1967; Bond et al. 1984; Kippenhahn & Weigert 1990; Fryer et al. 2001). For sufficiently massive stars, of mass $M \gtrsim 260M_\odot$, such unstable cores form a black hole, while for less massive cases, the outcome is a pair-unstable supernova (Fryer et al. 2001).

These examples illustrate that the gravitational collapses set in when the stability of progenitors changes, and hence, a star marginally stable against quasi-radial oscillations may be regarded as an initial condition of the collapse. Massive stars in nature are rapidly rotating in general (Bond et al. 1984). Thus, determining the quasi-radial stability for rotating stars is an important subject in astrophysics for elucidating a plausible initial condition for the stellar collapse.

In many cases, one simply assumes that progenitors of neutron stars or black holes start collapsing when their adiabatic index Γ decreases below a well-known critical

value, $4/3$, against a quasi-radial instability. The number $4/3$ is the correct value only when the progenitor is not rotating. If the progenitor is rotating, it is not the valid criterion, since angular momentum has the effect of stabilizing the star against gravitational collapse. In particular, the number $4/3$ would not be a good indicator of the instability for pair-unstable collapse, since the progenitor stars are likely to be rapidly rotating (Bond et al. 1984, Fryer et al. 2001).

According to a Newtonian theory, slowly rotating stars that satisfy the following relation are unstable against a quasi-radial oscillation (Tassoul 1978; see also §2):

$$3\Gamma - 4 - 2(3\Gamma - 5)\beta < 0. \quad (1)$$

Here,

$$\beta \equiv \frac{T}{W}, \quad (2)$$

and T and W are rotational kinetic and gravitational potential energies, respectively. In the following, we restrict our attention to the rigidly rotating case for simplicity. Based on our numerical analysis in Newtonian gravity, the maximum value of β for rigidly rotating stars with polytropic equations of state, which is achieved when the velocity at the equatorial surface is equal to the Kepler velocity (i.e., at the mass-shedding limits), is approximately written as

$$\beta_{\max} \approx 0.00902 + 0.124 \left(\Gamma - \frac{4}{3} \right). \quad (3)$$

Combining equations (1) and (3), we find that for $\Gamma \lesssim 1.328$, all the rigidly rotating stars are unstable against a quasi-radial oscillation in Newtonian gravity. However, even for $1.328 \lesssim \Gamma < 4/3$, the stars can be stabilized by the effect of rotation.

General relativistic gravity destabilizes the rapidly rotating stars even in the case of $\Gamma > 1.328$. One of the purposes of this paper is to numerically determine the criteria for the onset of the instability. In particular, we focus on the secular stability of rigidly rotating stars with soft equations of state in equilibrium against a quasi-radial oscillation. The stability is determined using the turning-point method (Friedman et al. 1988). To model the rotating stars, we adopt the polytropic equations of state with $n \gtrsim 3$ ($\Gamma \lesssim 4/3$).

From a general relativistic point of view, it is interesting to ask how large the nondimensional angular momentum parameter $q \equiv J/M^2$ can be for the marginally stable stars that may be plausible progenitors of gravitational collapse. Axisymmetric hydrodynamic simulations in general relativity (Nakamura 1981; Stark & Piran 1985; Piran & Stark 1986; Nakamura et al. 1987; Shibata 2000) have provided many numerical results that show that for $q > 1$, a black hole is not formed after the collapse. That is, the value of q may be a good indicator in predicting the final fate of the stellar collapse of massive objects. As Baumgarte and Shapiro (1999) indicate, the value of q for a marginally stable star with $\Gamma = 4/3$ is very close to unity. Cook et al. (1994) show that the critical value of q increases with decreasing of Γ for $4/3 < \Gamma \lesssim 5/3$. This implies that for $\Gamma < 4/3$, the critical value is likely to be larger than unity, even for the case of $4/3 - \Gamma \ll 1$. The second purpose of this paper is to confirm this fact.

The third purpose is to predict the outcomes of gravitational collapse of the marginally stable stars. As mentioned above and confirmed below, the value of q for the marginally stable stars with $\Gamma < 4/3$ can be larger than unity. This suggests that a rotating star with $\Gamma < 4/3$ of mass large enough collapse to a black hole in the nonrotating case may not form a black hole because of its significant effect of the angular momentum. In the analysis, we assume that (i) the collapse proceeds in an axisymmetric manner, (ii) the rest-mass distribution as a function of the specific angular momentum is preserved, and (iii) the pressure never halt the collapse. We indicate that the criterion for no black hole formation, $q > 1$, which has been suggested so far, is not very good for rigidly rotating stars with the soft equations of state. We predict that even for $1 < q \lesssim 2.5$, a black hole may be formed as a result of gravitational collapse of the rigidly rotating and marginally stable stars. However, for $q \gtrsim 2.5$, the significant angular momentum will prevent the collapsing stars from directly forming a black hole.

The paper is organized as follows. In §2, we present a semi-analytic calculation for determining the stability of rotating stars against a quasi-radial oscillation. In §3, the secular stability of rotating stars in general relativity is numerically determined. In §4, we predict the final outcomes of the stellar collapse for very massive stars, assuming that the initial condition is a marginally stable star, as determined in §3. Section 5 is devoted to a summary. Throughout this paper, the pressure is given by the polytropic equation of state as

$$P = K\rho^\Gamma, \quad \Gamma = 1 + \frac{1}{n}, \quad (4)$$

where P is the pressure, ρ the baryon rest-mass density, K the polytropic constant, and n the polytropic index, which

is chosen in the range between 3 and 3.05. We adopt the geometrical units $G = c = 1$, where G and c denote the gravitational constant and the speed of light, respectively.

2. SEMI-ANALYTIC EXPLORATION FOR STABILITY

To semi-analytically determine the marginally stable rotating stars with the polytropic equations of state with $n \gtrsim 3$, we follow the method described by Zel'dovich & Novikov (1971) and Shapiro & Teukolsky (1983) and write the total energy as the sum of the internal energy, the Newtonian potential energy, the rotational kinetic energy (in Newtonian order), and a post-Newtonian correction. These terms can be written

$$E = k_1 K M \rho_c^{1/n} - k_2 M^{5/3} \rho_c^{1/3} + k_5 J^2 M^{-5/3} \rho_c^{2/3} - k_4 M^{7/3} \rho_c^{2/3}, \quad (5)$$

where k_1 , k_2 , k_4 , and k_5 are structure constants that are constructed from Lane-Emden functions (Shapiro & Teukolsky 1983) and listed in Table 1. M is the total mass and ρ_c the central density.

To derive equation (5), we assume that the stars are spherical and rigidly rotating. In the treatment by Zel'dovich & Novikov (1971) and Shapiro & Teukolsky (1983), the authors take into account a nonsphericity, assuming that a spherical surface of constant density in the nonrotating case transforms into a spheroidal surface enclosing the same volume. This approximate method is likely to be good for stars with very stiff equations of state with $n \sim 0$. However, this is not the case for the stars with $n \gtrsim 3$, since the density profile is not as uniform as that for $n = 0$. Actually, in their approximate treatment, the value of β for given values of ρ_c and M is significantly overestimated for $n \sim 3$. In the case of the soft equations of state, the shape of the central region, around which most of the mass is concentrated, is nearly spherical even at mass-shedding limits. Therefore, we neglect the nonsphericity here. In this treatment, the value of J is considered an input quantity. Note also that the second post-Newtonian term is omitted in the present treatment, in contrast to those by Zel'dovich & Novikov (1971) and Baumgarte & Shapiro (1999). This is because the compactness of stable stars with a soft equation of state is very small, and hence, the second post-Newtonian correction produces a very small effect on the equilibrium and stability, as illustrated by Baumgarte & Shapiro (1999) for $n = 3$.

Taking the first derivative of equation (5) with respect to the central density yields a condition for equilibrium,

$$0 = \frac{\partial E}{\partial x} = \frac{3}{n} k_1 K M x^{(3-n)/n} - k_2 M^{5/3} + 2k_5 J^2 M^{-5/3} x - 2k_4 M^{7/3} x, \quad (6)$$

where $x \equiv \rho_c^{1/3}$. For a stable equilibrium, the second derivative of equation (5) has to be positive. Therefore, a root of the second derivative marks the onset of a quasi-radial instability:

$$0 = \frac{\partial^2 E}{\partial x^2} = \frac{3}{n} \left(\frac{3}{n} - 1 \right) k_1 K M x^{(3-2n)/n} + 2k_5 J^2 M^{-5/3} - 2k_4 M^{7/3}. \quad (7)$$

By combining equations (6) and (7), we derive a relation between M and x for the marginally stable stars as

$$M = \left[\frac{3(2n-3)k_1}{n^2k_2} K x^{(3-n)/n} \right]^{3/2}. \quad (8)$$

For $n = 3$, $MK^{-3/2}$ is a constant as 4.5548 irrespective of x (Baumgarte & Shapiro 1999). For $n > 3$, M decreases with increasing ρ_c for a fixed value of K , and in the limit $\rho_c \rightarrow 0$, $M \rightarrow \infty$.

From equations (7) and (8), J/M^2 for the marginally stable stars (defined as q_{mar}) can be written as a function of the density:

$$q_{\text{mar}}^2 = \frac{k_4}{k_5} + \frac{n^2(n-3)k_2^2}{6(2n-3)^2k_1k_5} (\rho_c K^n)^{-1/n}. \quad (9)$$

For $n = 3$, q_{mar} is a constant as 0.873 and for $n > 3$, q_{mar} decreases with increasing ρ_c . In the limit $\rho_c \rightarrow 0$, $q_{\text{mar}} \rightarrow \infty$. As we show in §3, the maximum value of ρ_c along the sequence of the marginally stable stars is reached at mass-shedding limits. At that point, q_{mar} reaches its minimum (q_{min}). Thus, the value of q_{mar} for the marginally stable stars is always larger than q_{min} for $n > 3$.

Using equation (6), equation (7) can be rewritten as

$$0 = 3\Gamma - 4 - 2(3\Gamma - 5) \left(\frac{k_5}{k_2} q^2 - \frac{k_4}{k_2} \right) M^{2/3} \rho_c^{1/3}. \quad (10)$$

In the present approach, the second term is related to $\beta(=T/W)$ by

$$\beta = \frac{k_5}{k_2} q^2 M^{2/3} \rho_c^{1/3}. \quad (11)$$

Using this relation, equation (10) is rewritten as

$$0 = 3\Gamma - 4 - 2(3\Gamma - 5) \left(\beta - \frac{k_4}{k_2} M^{2/3} \rho_c^{1/3} \right). \quad (12)$$

The term $M^{2/3} \rho_c^{1/3}$ is related to the compactness parameter M/R_s , where R_s is the radius of the spherical star, as follows:

$$\frac{M}{R_s} = \alpha M^{2/3} \rho_c^{1/3}. \quad (13)$$

Here, the values of α are determined by the Lane-Emden function and listed in Table 1. Using this relation, equation (12) is written as

$$0 = 3\Gamma - 4 - 2(3\Gamma - 5) \beta - \alpha' \frac{M}{R_s}, \quad (14)$$

where $\alpha' = 2(5 - 3\Gamma)k_4/(\alpha k_2)$ which is in the range between 6.747 and 7.137 for $\Gamma = 4/3$ –1.328. Equation (14) is a familiar formula often presented in the standard textbooks such as Tassoul (1978). For the case in which $\beta = 0$, the equation agrees with the formula derived by Chandrasekhar (1964) for $n = 3$.

3. NUMERICAL ANALYSIS FOR STABILITY

In §2, we determined sequences of the marginally stable stars for $n \gtrsim 3$ ($\Gamma \lesssim 4/3$) in terms of a semi-analytic calculation. As shown in this section, they indeed give good

approximate sequences for the marginally stable rotating stars. However, they are not exact sequences after all. To determine them precisely, numerical computations are necessary.

Another drawback in the semi-analytic calculation is that the maximum value of the angular velocity for a given value of J is not determined precisely. For rigidly rotating stars, the velocity should be smaller than the Kepler velocity at the equatorial surface. This restricts the allowed region for the rigidly rotating stars: A sequence of stars at mass-shedding limits divides the regions where the rigidly rotating stars exist and where they do not. To determine the sequence at the mass-shedding limits, the configuration of the stars has to be accurately computed. Thus, numerical computation is inevitable.

3.1. Basic equations

To study the secular stability of rotating stars against a quasi-radial oscillation, the equilibrium solutions in general relativity are computed. We write the energy momentum tensor of the Einstein equation for an ideal fluid as

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}, \quad (15)$$

where u^μ is the four velocity, $h \equiv 1 + \varepsilon + P/\rho$ is the enthalpy, ε is the specific internal energy, and $g^{\mu\nu}$ is the spacetime metric. As in §2, the polytropic equations of state are adopted. Using the first law of the thermodynamics, the specific internal energy ε in the polytropic equations of state is written as

$$\varepsilon = \frac{nP}{\rho}. \quad (16)$$

As in §2, we pay attention only to rigidly rotating stars setting the angular velocity $\Omega \equiv u^\varphi/u^t$ as a constant.

With the polytropic equation of state, physical units enter the problem only through the polytropic constant K , which can be chosen arbitrarily or else completely scaled out of the problem. Indeed, $K^{n/2}$ has units of length, time and mass, and K^{-n} has units of density in the geometrical units. Thus, in the following, we only show non-dimensional quantities, which are rescaled by K . In other words, we adopt the units with $c = G = K = 1$.

Following Butterworth & Ipser (1976), the line element is written as

$$ds^2 = -e^{2\nu} dt^2 + B^2 e^{-2\nu} r^2 \sin^2 \theta (d\varphi - \omega dt)^2 + e^{2\zeta - 2\nu} (dr^2 + r^2 d\theta^2), \quad (17)$$

where ν , B , ω , and ζ are field functions. The first three obey elliptic-type equations in axial symmetry, and the last one an ordinary differential equation. These equations are solved using the same method as that described by Shibata & Sasaki (1998).

The total baryon rest-mass M_* , Komar mass (gravitational mass) M , proper mass M_p , Komar angular momentum J , rotational kinetic energy T , and gravitational potential energy W are defined from the energy momentum

tensor or matter variables as

$$M_* = 2\pi \int \rho u^t B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta), \quad (18)$$

$$M = 2\pi \int (-2T_t^t + T_\mu^\mu) B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta), \quad (19)$$

$$M_p = 2\pi \int \rho u^t (1 + \varepsilon) B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta), \quad (20)$$

$$J = 2\pi \int \rho h u^t u_\varphi B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta), \quad (21)$$

$$T = \frac{1}{2} J \Omega, \quad (22)$$

$$W = M_p - M + T, \quad (23)$$

where $W > 0$. From these quantities, the well-known nondimensional parameters are defined as $\beta \equiv T/W$ and $q \equiv J/M^2$. The ADM mass M_{ADM} is defined from the asymptotic behavior of ν as

$$M_{\text{ADM}} = - \lim_{r \rightarrow \infty} \nu r. \quad (24)$$

For the stationary spacetime, M_{ADM} is equal to M (Beig 1978). Thus, accuracy of the numerical solutions can be measured checking the relation $M = M_{\text{ADM}}$.

In addition to these quantities, we often refer to the central density, ρ_c , which is used to specify a rotating star for a given set of β and M , and to the equatorial circumferential radius, R , by which a compactness parameter is defined as M/R .

3.2. Analysis for secular stability

The secular stability for rigidly rotating stars against quasi-radial oscillations can be determined by a turning-point method as established by Friedman et al. (1988) and subsequently used by Cook et al. (1992, 1994). According to the turning-point theorem, a change of the sign of $dM/d\rho_c$ along a curve of a constant value of J indicates the change of a secular stability. Thus, in the numerical computation, curves of constant values of J are computed and plotted in the plane composed of M and ρ_c for determining the turning points.

To determine the region for the stable stars, in the present case, one should compute $dM/d\rho_c$ along curves of a constant value of J . For $(dM/d\rho_c)_J > 0$ (< 0), the stars are stable (unstable) against gravitational collapse. From this fact, one can distinguish the stable region from the unstable one.

3.3. Numerical results

Numerical computation was carried out using the spherical polar coordinates (r, θ, φ) . For a solution of elliptic-type equations in axial symmetry, a uniform grid for r and $\cos \theta$ is adopted with the typical grid size $(N_r, N_\theta) = (1000, 160)$. To investigate convergence for the location of the turning points, the computations were also performed with $N_r = 500, 750$, and 1500 and $N_\theta = 100$. We found that the location depends very weakly on N_θ as long as it is larger than 100. On the other hand, the convergence of the location is slow with increasing N_r (cf. Figure 1). However, with $N_r \gtrsim 1000$, the convergence is good, except for the region near mass-shedding limits.

In the typical computations, the equatorial radius of a star is covered by $0.6N_r$ grid points. In our method, the outer boundaries are located at a finite radius, and thus, the outer boundary conditions are approximate. However, the stellar equatorial radius is larger than several hundred M for stable rotating stars with $n \geq 3$ (cf. Table 2), implying that the radius of the outer boundaries is larger than $\sim 1000M$. In this setting, even the approximate condition is very accurate. To check the magnitude of the numerical error associated with this approximation, we performed several computations changing the location of the outer boundaries with a fixed grid spacing, and indeed found that the errors in mass, density, and angular momentum are much smaller than 1%.

Figure 1 is a summary of the numerical results for $\Gamma = 1.328, 1.329, 1.330, 1.332, 1.333$, and $4/3$. In each panel, the dashed curve denotes the sequence of rotating stars at mass-shedding limits. Namely, there is no rigidly rotating star in the right-hand side of these curves. The solid curve denotes the sequences of constant values of J . The thick solid curve, the crosses, the solid squares, and the open circles with the dotted curve are sequences of the marginally stable stars with $N_r = 1000, 500, 750$, and 1500 , respectively. It is found that there is a critical value of $J = J_{\text{min}}$, where J_{min} denotes a constant, below which all the stars are unstable. The values of J_{min} are determined at the intersection between the curves for the sequence of the mass-shedding limit and for the sequence of the turning points (cf. Table 2). In the following, we refer to such a point as an intersection point.

Here, we write the density of the marginally stable stars as $\rho_t(J)$ for $J \geq J_{\text{min}}$. From the fact that all the nonrotating stars are unstable for $\Gamma \leq 4/3$, we can determine that with $\rho < \rho_t$ for a given value of $J \geq J_{\text{min}}$, the stars are unstable and otherwise, they are stable. Therefore, the rotating stars located between the sequences of the mass-shedding limits and of the turning points are stable against quasi-radial oscillations for $\Gamma < 4/3$. For $\Gamma = 4/3$, stars with $\rho > \rho_t$ for a given value of J are unstable. Thus, the stars located in the higher-mass side of the sequence of the turning points are stable.

The long-dashed curves denote the relation of equation (8) with $K = 1$; the sequence of the marginally stable stars derived by the semi-analytic calculation in §2. It is found that the long-dashed curves agree approximately with the numerical sequences of the turning points for $\Gamma \geq 1.329$. The error in mass for a given value of the central density is typically 0.1%. This indicates that the semi-analytic calculation gives a good approximate solution for the marginally stable stars.

For $\Gamma = 1.328$, the long-dashed curve is located in the right-hand side (the higher-density side) of the dashed curve, where there are no rigidly rotating stars. Since in the semi-analytic calculation the configuration of the stars cannot be determined, unrealistic solutions for marginally stable stars may be derived. Thus, the long-dashed curve denotes the unrealistic sequence of the marginally stable stars and indicates that there are no stable stars for $\Gamma = 1.328$.

The intersection point denotes the most compact marginally stable star for a given equation of state. Unfortunately, it is not easy to identify the turning points accurately near the mass-shedding limits. This fact can be also

recognized from the fact that convergence of the numerical solutions for the marginally stable stars with increasing N_r is achieved more slowly near the mass-shedding sequence than for other solutions.

Figure 1 shows that the numerical sequences of the turning points appear to bend sharply near the mass-shedding limits irrespective of the value of Γ . However, such tendency seems to be spurious because with increasing N_r , the point of the bending tends to approach the mass-shedding sequences and eventually disappear for $N_r \rightarrow \infty$. This illustrates that the intersection point is not accurately determined from the raw data sets for $N_r \leq 1500$. In the present work, for the determination of the intersection point, we pick up the data sets of the turning points slightly far away from the mass-shedding sequences, and perform an extrapolation of the turning points to the mass-shedding limits. The several quantities for the intersection point determined by this method are listed in Table 2. We estimate that the errors of the values for M , J , ρ_c , and β are $\sim 0.1\%$, 1% , 10% , and 0.1% , respectively.

Note that our results for $n = 3$ slightly disagree with those by Baumgarte & Shapiro (1999). According to our results, the values of ρ_c and q at the intersection point are $\approx 1.1 \times 10^{-8}$ and 0.91 , while their results are $\approx 0.7 \times 10^{-8}$ and 0.97 , respectively. This disagreement comes partly from numerical errors contained in our results of magnitude $\sim 10\%$ for ρ_c and $\sim 1\%$ for q . However, even if this error is taken into account, the results of the two groups disagree. Currently, the reason is not very clear. Here, we point out a fact: In the limit $\rho_c \rightarrow 0$, the mass of the marginally stable stars from our results converges to ≈ 4.555 , which agrees with the exact value, the value of the Newtonian polytrope with $n = 3$. On the other hand, the asymptotic value of their results slightly disagrees with the exact value in this limit. Thus, as far as the results for $\rho_c \rightarrow 0$ are concerned, our numerical results appear to be more accurate.

Figure 1 shows that for the smaller value of Γ , the allowed region for the stable stars becomes narrower. We find that for $\Gamma \leq 1.328$, there is no region for the stable stars and that for $\Gamma = 1.329$, the region for the stable stars is very narrow. From these results, we conclude that all the rigidly rotating stars are unstable for $\Gamma \lesssim 1.329$.

According to a semi-analytic estimate presented in §2, marginally stable stars satisfy equation (12). In this relation, the terms $3\Gamma - 4$, $2(3\Gamma - 5)\beta$, and $2(3\Gamma - 5)k_4 k_2^{-1} M^{2/3} \rho_c^{1/3}$ are of nearly identical order as $\sim 10^{-2}$ (except for the cases $\Gamma \approx 4/3$ in which $3\Gamma - 4 \approx 0$). To confirm the validity of this relation, we calculated the relative error defined as

$$Q \equiv \frac{1}{2(3\Gamma - 5)\beta} \left[3\Gamma - 4 - 2(3\Gamma - 5) \left(\beta - \frac{k_4}{k_2} M^{2/3} \rho_c^{1/3} \right) \right]. \quad (25)$$

Figure 2 shows the values of Q as a function of J for $\Gamma = 1.329, 1.330, 1.332$, and 1.333 . It is found that $|Q|$ is less than $\sim 2\%$ irrespective of Γ . This indicates fair validity of the relation (12).

For $\Gamma = 1.329$ – 1.332 , the value of Q appears to asymptotically approach a constant with increasing J (with decreasing ρ_c). This is reasonable, because even in the limit $\rho_c \rightarrow 0$, a marginally stable star is rotating, and hence, the semi-analytic calculation in which the stars are assumed to

be spherical contains certain systematic error. Therefore, the asymptotic value of Q may be regarded as a typical magnitude of the systematic error associated with neglect of the nonspherical deformation in the semi-analytic calculation.

Equation (9) implies that if q_{\min} is larger than unity, the value of q for all the marginally stable stars is larger than unity. As shown above and in Table 2, q_{\min} is larger than unity for $1.329 \lesssim \Gamma \lesssim 1.332$, implying that the value of q is always so for the rigidly rotating and marginally stable stars.

4. PREDICTING THE FINAL OUTCOME

The marginally stable stars determined in §3 may be plausible approximate initial conditions for a stellar core collapse or a pair-unstable collapse. Here, we predict the outcome of the collapse paying particular attention to the black hole formation case under the assumptions that (i) the collapse proceeds in an axisymmetric manner, (ii) the viscous angular momentum transport during the collapse is negligible, and (iii) the pressure and heating effect never halt the collapse. Actually, in a stellar core collapse with mass larger than $\sim 40M_\odot$ (Fryer 1999) and in a pair-unstable collapse with mass larger than $\sim 260M_\odot$ (Fryer et al. 2001), the pressure support and the nuclear burning energy are so small that the final outcome is likely to be a black hole. The numerical analysis is carried out in the same manner as that of Shibata & Shapiro (2002).

Since viscosity is assumed to be negligible during the collapse, the specific angular momentum j of each fluid element is conserved in an axisymmetric system. Here, j is defined as

$$j \equiv hu_\varphi. \quad (26)$$

Next, we define rest-mass distribution $m_*(j)$ as a function of j , which is the integrated baryon rest-mass of all fluid elements with specific angular momentum less than a given value j_0 :

$$m_*(j_0) \equiv 2\pi \int_{j < j_0} \rho u^t B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta). \quad (27)$$

Let us assume that a seed black hole is formed during the collapse and consider the innermost stable circular orbit (ISCO) around the growing black hole at the center. If j of a fluid element is smaller than the value at the ISCO (j_{ISCO}), the element will fall into the black hole eventually. Now the possibility exists that some fluid can be captured even for $j > j_{\text{ISCO}}$, if it is in a noncircular orbit. Ignoring these trajectories yields the minimum amount of mass that will fall into the black hole at each moment. The value of j_{ISCO} changes as the black hole grows. If j_{ISCO} increases, additional mass will fall into the black hole. However, if j_{ISCO} decreases, ambient fluid will no longer be captured. This expectation suggests that when j_{ISCO} reaches a maximum value, the dynamical growth of the black hole will have already terminated (i.e., before the maximum, the point for $j = j_{\text{ISCO}}$ will be reached). As Shibata & Shapiro (2002) demonstrated, prediction of the growth of the mass and spin by this method is in good agreement with a numerical result in the collapse with $n = 3$, suggesting that this is also a good method for $n \gtrsim 3$.

To analyze the growth of the black hole mass, we generate Figures 3 and 4. Figure 3 shows $m_*(j)/M_*$ as a func-

tion of j/M_* for $\Gamma = 1.329, 1.330, 1.332$, and 1.333 . Here, we choose the marginally stable rotating stars at mass-shedding limits. That is, we choose the most compact and most rapidly rotating, marginally stable stars.

Each panel (i) in Figures 4 (a)–(d) denotes $q_* \equiv J(j)/m_*(j)^2$ as a function of $m_*(j)/M_*$. Here, $J(j)$ denotes the total angular momentum with the specific angular momentum less than a given value j_0 and is defined according to

$$J(j_0) = 2\pi \int_{j < j_0} \rho h u^t u_\varphi B e^{2\zeta - 2\nu} r^2 dr d(\cos \theta). \quad (28)$$

Now, $J(j)/m_*(j)^2$ and $m_*(j)$ may be approximately regarded as the instantaneous spin parameter and mass of a black hole, since the baryon rest-mass is nearly equal to the gravitational mass for the soft equations of state. Therefore, the solid curve in each panel (i) of Figure 4 may be interpreted as an approximate evolutionary track for the angular momentum parameter of the growing black hole. It indicates that with increasing the black hole mass, the spin parameter also increases.

If we assume that $m_*(j)$ and q_* are the instantaneous mass and spin parameter of the growing black hole and that the spacetime can be approximated instantaneously by a Kerr metric, we can compute j_{ISCO} (Bardeen et al. 1972; chapter 12 of Shapiro & Teukolsky, 1983). In each panel (ii) of Figure 4, we show $j_{\text{ISCO}}[m_*(j), q_*(j)]$ as a function of $m_*(j)$. The maximum of j_{ISCO} is reached at $m_*(j) = m_{*\text{crit}}$, where $m_{*\text{crit}}/M_* \approx 0.57, 0.77, 0.89$, and 0.92 for $\Gamma = 1.329, 1.330, 1.332$, and 1.333 , respectively (Fig. 4, *circles*). The value of j_{ISCO} is slightly smaller than j at $m_*(j) = m_{*\text{crit}}$: The point for $j = j_{\text{ISCO}}$ (Fig. 4, *triangles*) is reached at a value of $m_*(j)$, which is only slightly smaller than $m_{*\text{crit}}$. This suggests that the mass of the black hole will increase to $\sim m_{*\text{crit}}$. However, after the maximum of j_{ISCO} is reached, $j_{\text{ISCO}}/m_*(j)$ steeply decreases above this mass fraction. Thus, once the black hole reaches the point of $m_*(j) = m_{*\text{crit}}$, it will stop growing dynamically. Panels (ii) of Figure 4 shows that at this stage, $q_* \approx 0.94, 0.87, 0.77$, and 0.73 for $\Gamma = 1.329, 1.330, 1.332$, and 1.333 , respectively.

It is interesting to note that for $\Gamma = 1.329$ – 1.332 , the total value of q for the system is larger than unity. However, as far as the inner region of the collapsing stars is concerned, it is much smaller than unity and, hence, a seed black hole is likely to be formed. The large value of $q > 1$ will be reflected in the large mass fraction of formed disks surrounding the central black hole. For the cases with $\Gamma = 1.329, 1.330$, and 1.332 , the fraction of the disk mass will be $\sim 40\%, 20\%$, and 10% , respectively. The present numerical analysis indicates that the global value of q is not always a good indicator for predicting the final outcome of stellar collapse with soft equations of state. However, the conclusion drawn here is based on several assumptions. To confirm it more strictly, fully general relativistic simulations are necessary.

After the dynamical collapse, a system of a black hole and surrounding disks in a nearly quasiequilibrium state will be formed. Subsequent evolution of the system is determined by the viscous time scale of the accretion disks. Since a large fraction of the total mass forms disks that eventually fall into the central black hole, the black hole

will spin up to $q \lesssim 1$ for $\Gamma \lesssim 1.332$.

Next, we pay attention to the fate of the collapse for general marginally-stable stars. For the analysis, we only need to repeat the same procedure described above. Fortunately, without detailed numerical computations, it is possible to approximately predict the outcome using a semi-analytic calculation as follows.

In the semi-analytic method adopted in §2, the mass and angular momentum contained inside a cylinder of radius ϖ are written as

$$M(\varpi) = M\mathcal{M}(y), \quad (29)$$

$$J(\varpi) = J\mathcal{J}(y), \quad (30)$$

where $y \equiv \varpi \xi_1 / R_s$ and ξ_1 denotes the Lane-Emden radial coordinate of the stellar surface (e.g., Shapiro & Teukolsky 1983). \mathcal{M} and \mathcal{J} are computed from the Lane-Emden functions as

$$\mathcal{M}(y) = \frac{\xi_1}{\theta'_1} \int_0^y \theta^n \xi^2 (1 - \cos t) d\xi, \quad (31)$$

$$\mathcal{J}(y) = \frac{\xi_1}{\kappa \theta'_1} \int_0^y \theta^n \xi^4 \left(1 - \cos t - \frac{1 - \cos^3 t}{3} \right) d\xi, \quad (32)$$

where $t = \sin^{-1}(\min[1, \varpi \xi_1 / (R_s \xi)])$, θ'_1 denotes $|d\theta/d\xi|$ at $\xi = \xi_1$, and

$$\kappa \equiv \frac{2}{3\xi_1^4 \theta'_1} \int_0^{\xi_1} \theta^n \xi^4 d\xi. \quad (33)$$

Now we define the nondimensional angular momentum parameter estimated for the fluid elements inside the cylinder of radius ϖ as

$$q(\varpi) \equiv \frac{J(\varpi)}{M(\varpi)^2} = q \frac{\mathcal{J}}{\mathcal{M}^2}. \quad (34)$$

Thus, a curve of $q(\varpi)$ is determined for a given value of q in this model.

To verify that this semi-analytic estimate for $q(\varpi)$ is a good approximation, we plot $q(\varpi)$ as a function of \mathcal{M} in each panel (i) of Figure 4 (dotted curves). To plot these curves, we choose the same value of q for each corresponding numerical model presented in Figure 4. Note that the specific angular momentum is written as $\varpi^2 \Omega$ in this semi-analytic model. Thus, $M(\varpi)$ may be regarded as the mass distribution as a function of the specific angular momentum.

Comparison between the dotted and solid curves in Figure 4 confirms that the numerical and semi-analytic results are in good agreement. This indicates that the semi-analytic method adopted here is appropriate for an approximately quantitative study.

According to equation (34), the curves of the semi-analytic results shift upward with increasing q . For a sufficiently large value of q , the value of $q(\varpi = 0)$ exceeds unity and, therefore, $q(\varpi) > 1$ for any value of ϖ . Such large values of q are achieved for marginally stable stars of small compactness. In this case, it is natural to expect that even a seed black hole will not be formed in the gravitational collapse. Thus, the collapse of marginally stable stars with $1.329 \leq \Gamma < 4/3$ and with a small compactness will not result directly in a black hole but in a disk or a torus. The critical value of q above which a black hole

will not be formed (q_{BH}) is ~ 2.5 irrespective of Γ (see the solid and dotted curves for $q = 2.517$ in Figure 4(a) for an example).

Using equations (8), (9), and (13), the compactness of the marginally stable stars can be written as a function of q :

$$\frac{M}{R_s} = \frac{k_2(3-n)\alpha}{2k_5(3-2n)(q^2 - k_4/k_5)}. \quad (35)$$

For $q = 2.5$, $M/R_s = 2.61 \times 10^{-4}$, 2.02×10^{-4} , 8.18×10^{-5} , and 2.06×10^{-5} for $\Gamma = 1.329$, 1.330 , 1.332 , and 1.333 , respectively. In the iron core collapse of massive stars, the radius and mass before the collapse are \sim a few thousand kilometer and at most a few M_\odot , respectively (Umeda & Nomoto, unpublished). Thus, the compactness is $\sim 10^{-3}$ and, hence, the value of q at the onset of the instability is likely to be smaller than 2.5 if the star is assumed to be rigidly rotating. On the other hand, in the pair-unstable collapse, the density and mass at the onset of the collapse are $\sim 10^4$ g/cm³ and $100M_\odot$ (Bond et al. 1984). This implies that the compactness is $\sim 2 \times 10^{-4}$. At the onset of a pair-unstable collapse, the adiabatic index of the star decreases from $\sim 4/3$ to smaller value on a time scale much longer than the dynamical time scale (Bond et al. 1984). If the star is rotating sufficiently rapidly, the instability will set in when Γ decreases below ~ 1.329 . For $\Gamma = 1.329$ with $M/R_s \sim 2 \times 10^{-4}$, the value of q at the onset of the instability is larger than 2.5 if the star is rigidly rotating. This suggests that the collapse may not lead to a black hole directly in this case.

A collapse with $q > q_{\text{BH}}$ will result in formation of a disk or a torus, which will be subsequently unstable against nonaxisymmetric deformation. After the nonaxisymmetric instabilities turn on, angular momentum will be transported from the inner region to the outer region, decreasing the value of q around the inner region below unity and resulting in the formation of a seed black hole.

To clarify whether the above scenario is correct, it is obviously necessary to perform a three-dimensional numerical simulation in general relativity (e.g., Shibata 1999; Shibata & Uryū 2000, 2002; Font et al. 2002; Shibata et al. 2003). During the collapse, the typical length scale may change by a factor of 10^4 from $R_s/M \sim 10^4$ to 1. To follow the collapse by numerical simulation, very large computational resources will be necessary and, thus, the simulation for this phenomenon will be one of the computational challenges in the field of numerical relativity.

Finally, we note the following point. As shown above, $q(\varpi)$ near the rotational axis is smaller than the global value of q by a factor of 2 in the rigidly rotating case. This is the reason that the relation $q > 1$ is not likely to be a good criterion for no black hole formation. However, in the differentially rotating case or for stiffer equations of state with $\Gamma \gtrsim 2$, the ratio $q/q(\varpi = 0)$ becomes smaller than 2. For $q/q(\varpi = 0) \gtrsim 1$, the global value of q may be still an approximate indicator for predicting the outcome. Previous axisymmetric simulations in general relativity have not been performed taking an equilibrium with $\Gamma \sim 4/3$ in the rigidly rotating initial condition (Nakamura 1981; Stark & Piran 1985; Piran & Stark 1986; Nakamura et al. 1987; Shibata 2000). This is the reason why the previous numerical works have suggested that the value of q is a good indicator for predicting the outcome.

5. SUMMARY

We have investigated the secular stability of rigidly rotating stars against a quasi-radial oscillation in general relativity. It is found that all the rigidly rotating stars with $\Gamma \leq 1.328$ are unstable against a quasi-radial oscillation. Stars with $1.329 \lesssim \Gamma < 4/3$ that are unstable for the spherical case can be stabilized by the effect of rotation. Therefore, for rapidly rotating stars for which the adiabatic index gradually decreases from $\sim 4/3$ to smaller values, the instability against gravitational collapse will set in when the value of Γ decreases below ≈ 1.329 (not $4/3$).

The marginally stable stars for $\Gamma = 1.329$ – $4/3$ have been determined numerically. It is found that for $1.329 \leq \Gamma \leq 1.332$, the nondimensional angular momentum parameter q for *all* rigidly rotating and marginally stable stars is larger than unity. The value of q for the marginally stable stars increases with decrease of the compactness. Therefore, for a sufficiently small compactness, the value of q exceeds unity even for $1.332 < \Gamma < 4/3$. These results are in good agreement with those derived by a simple semi-analytic calculation presented in §2.

The final outcomes of axisymmetric collapse of the rigidly rotating and marginally stable stars are predicted assuming that the pressure never halts the collapse and that the mass distribution as a function of the specific angular momentum is preserved. It is found that even for the case $q > 1$, a black hole will form as a result of the gravitational collapse if the value of q is smaller than ~ 2.5 . This is due to the fact that the angular momentum in the central region of the rigidly rotating stars is not large enough and that the effective value of the nondimensional angular momentum parameter q for the inner region is smaller than unity. The outcome of the collapse in such cases will be a rapidly rotating black hole surrounded by a massive disk. For $q \gtrsim 2.5$, even the central region has angular momentum large enough to prevent collapsing to a black hole. In this case, the outcome will be a disk or a torus. It will be subsequently unstable against nonaxisymmetric deformation. Once the nonaxisymmetric structure is developed, the angular momentum will be transported outward by the nonaxisymmetric effects, decreasing the value of q in the inner region below unity and eventually forming a seed black hole.

Although the conjecture mentioned above is reasonable, the final fate of the collapse of marginally stable stars with $q \gtrsim 1$ can be determined only by fully general relativistic simulations. In particular for $q \gtrsim 2.5$, three-dimensional simulations are necessary since the collapsing star is likely to become unstable against nonaxisymmetric deformation. Since the length scale will change by a factor of 10^4 during the collapse, huge computational resources will be necessary for the simulation. The simulation for this problem will be one of the computational challenges in the field of numerical relativity.

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TABLE 1
STRUCTURE CONSTANTS OF SPHERICAL POLYTROPES

Γ	k_1	k_2	k_4	k_5	α
1.328	1.7877	0.63546	0.92247	1.1837	0.41330
1.329	1.7816	0.63613	0.92169	1.1876	0.41571
1.330	1.7756	0.63680	0.92091	1.1914	0.41810
1.332	1.7637	0.63812	0.91934	1.1991	0.42285
1.333	1.7578	0.63878	0.91856	1.2028	0.42522
4/3	1.7558	0.63900	0.91829	1.2041	0.42600

NOTE.—Values of k_i and α are computed from the Lane-Emden function.

TABLE 2
PARAMETERS OF MARGINALLY STABLE STARS AT MASS-SHEDDING LIMITS

Γ	M	J	q	β	R/M	ρ_c
1.329	5.435	52.0	1.76	0.00847	2.2e3	1.3e-10
1.330	5.179	37.0	1.38	0.00856	1.4e3	6.3e-10
1.332	4.779	23.6	1.03	0.00873	7.2e2	4.5e-9
1.333	4.613	20.0	0.940	0.00883	6.0e2	8.8e-9
4/3	4.560	19.0	0.914	0.00886	5.6e2	1.1e-8

NOTE.—Adiabatic index, M , J , $q = J/M^2$, $\beta = T/|W|$, R/M , and ρ_c of marginally stable stars at mass-shedding limits are shown with the units $c = G = K = 1$. Note that M_* is nearly equal to M in all the cases (the relative difference is smaller than 10^{-4}).

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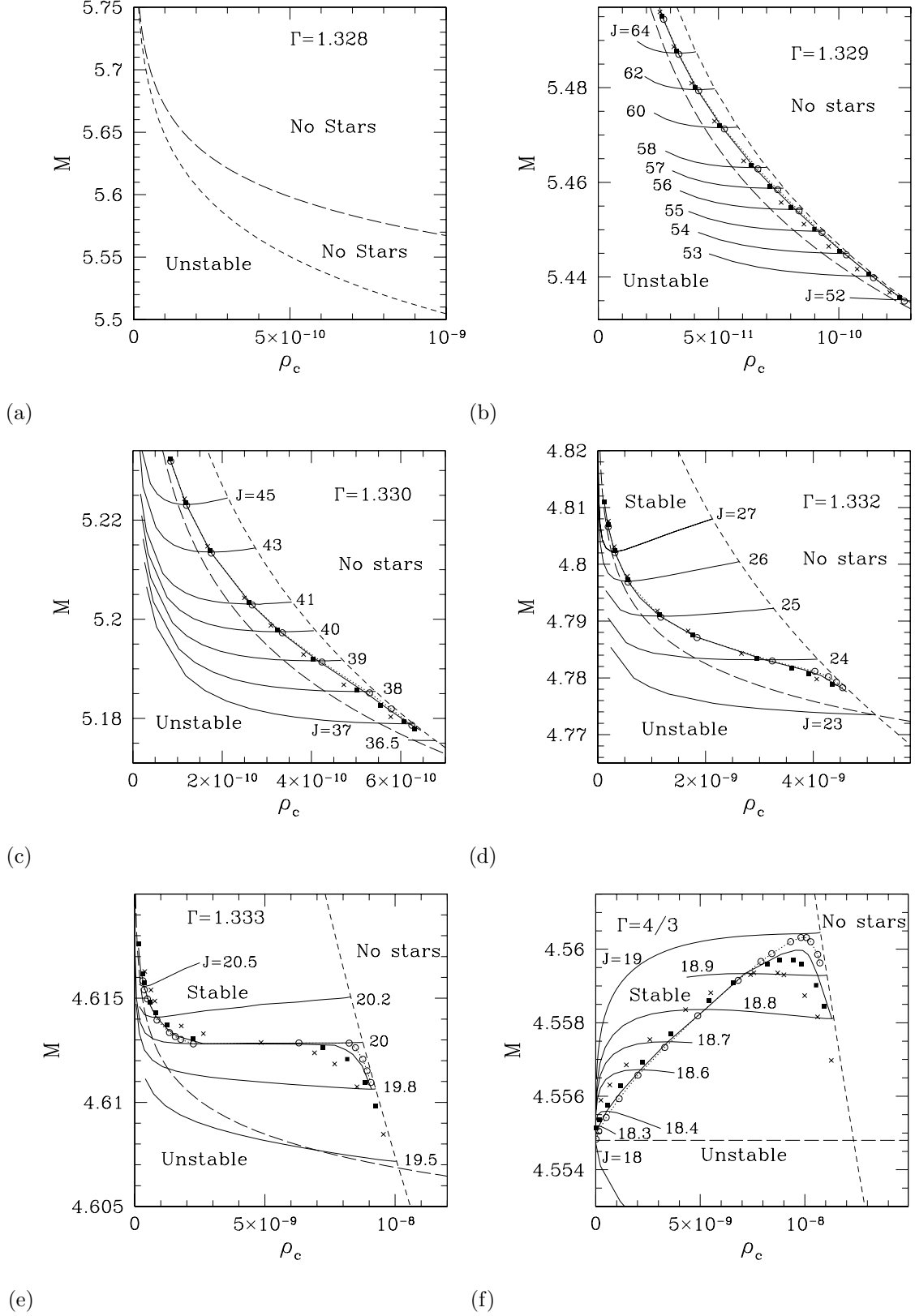


FIG. 1.— Gravitational mass M as a function of ρ_c for fixed values of J (thin solid curves), for sequences of rotating stars at mass-shedding limits (dashed curves), and for sequences of marginally stable stars (thick solid curves) for $\Gamma = 1.328, 1.329, 1.330, 1.332, 1.333$, and $4/3$ with $N_r = 1000$. Rigidly rotating stars are located on the left-hand side of the dashed curves, and thus, for $\Gamma = 1.328$, all the rigidly rotating stars are unstable. The long-dashed curves denote the sequences of the marginally stable stars derived by a semi-analytic calculation in §2. The crosses, filled squares, and the dotted curves with the open circles denote sequences of the marginally stable stars computed with $N_r = 500, 750$, and 1500 , respectively.

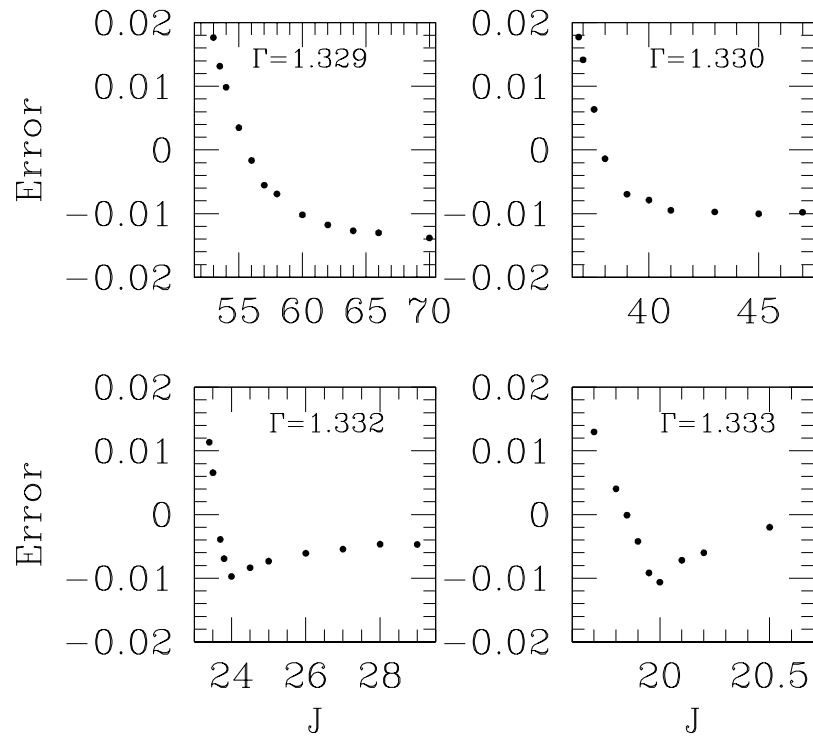


FIG. 2.— Relative error Q for marginally stable stars defined in equation (25) as a function of J for $\Gamma = 1.329, 1.330, 1.332$, and 1.333 .

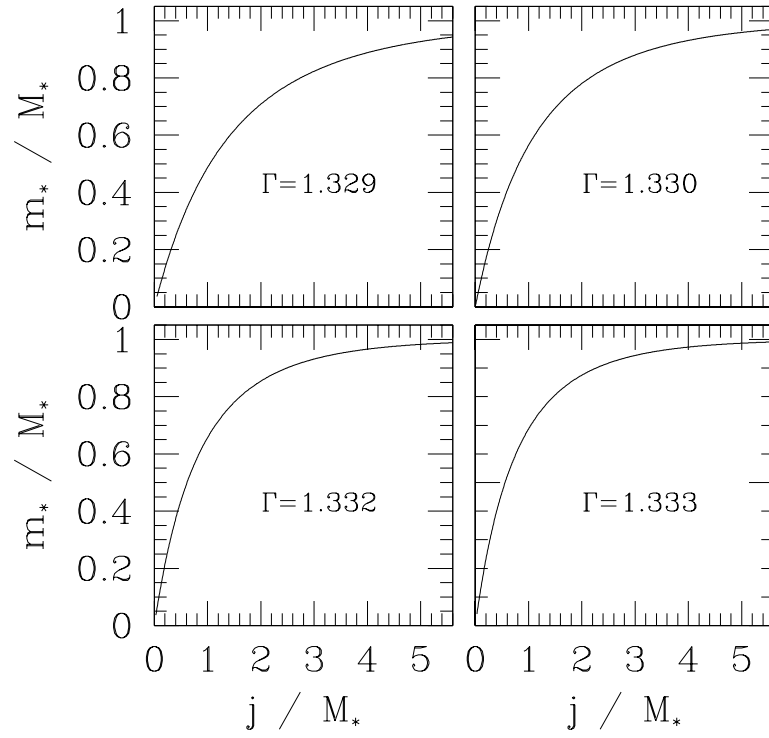


FIG. 3.— Fraction of the rest mass $m_*(j)/M_*$ as a function of j/M_* for the marginally stable stars at mass-shedding limits with $\Gamma = 1.329$, 1.330, 1.332, and 1.333. ($J = 52.0$, 37.0, 23.6, and 20.0, respectively; see Figure 1.)

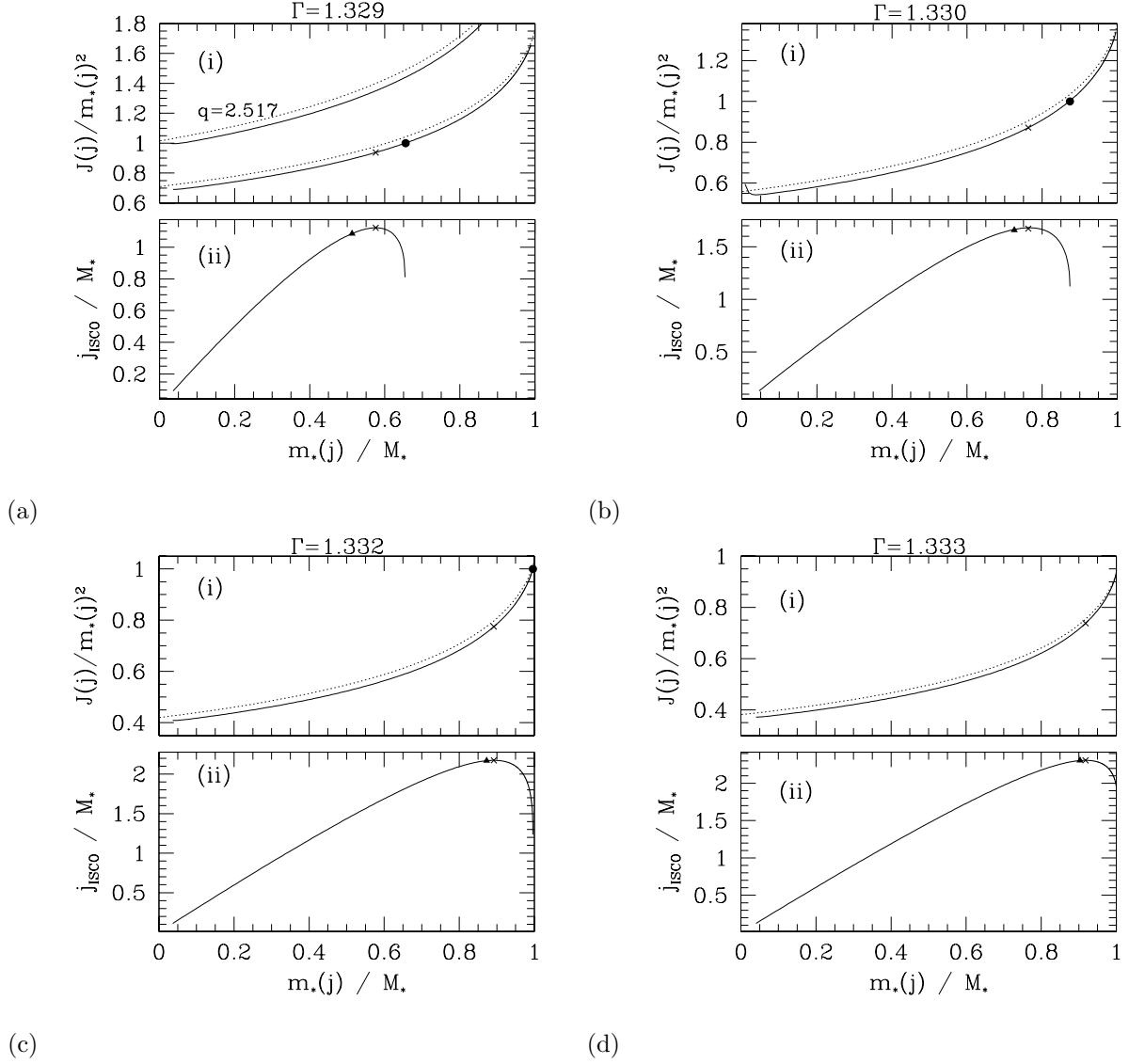


FIG. 4.— $q_* \equiv J(j)/m_*(j)^2$ and j_{ISCO}/M_* as a function of $m_*(j)/M_*$ for rigidly rotating stars at mass-shedding limits near the marginally stable criterion. (a) $\Gamma = 1.329$ and $J = 52$ ($q = 1.76$), (b) $\Gamma = 1.330$ and $J = 37.0$ ($q = 1.38$), (c) $\Gamma = 1.332$ and $J = 23.6$ ($q = 1.03$), and (d) $\Gamma = 1.333$ and $J = 20.0$ ($q = 0.94$). The crosses denote the point at which j_{ISCO} reaches maximum, the filled triangles the point at which $j = j_{\text{ISCO}}$, and the filled circles the point at which $q_* = 1$. The dotted curves denote the semi-analytic results for the same value of q as in the numerical results. For $\Gamma = 1.329$, the numerical and semi-analytic results for $q = 2.517$, for which $q(\varpi) \geq 1$ and hence a black hole will not be formed, are also plotted.